

What make them all so turbulent

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Abstract

We give a unified proof of the existence of turbulence for some classes of continuous interval maps which include, among other things, maps with periodic points of odd periods > 1 , some maps with dense chain recurrent points and densely chaotic maps.

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Let I be a compact interval in the real line and let $f : I \rightarrow I$ be a continuous map. It is well-known that [1, 2, 3, 4, 5] if (a) there exist a point c and an odd integer $n > 1$ such that $f^n(c) \leq c < f(c)$ or $f(c) < c \leq f^n(c)$, or (b) f has dense periodic points and $f^2(a) \neq a$ for some point a , or (c) there is a point whose ω -limit set with respect to f contains a fixed point z of f and a point $\neq z$, or (d) f is densely chaotic, i.e., the set $LY(f) = \{(x, y) \in I \times I : \limsup_{n \rightarrow \infty} |f^n(x) - f^n(y)| > 0 \text{ and } \liminf_{n \rightarrow \infty} |f^n(x) - f^n(y)| = 0\}$ is dense in $I \times I$, then f^2 is turbulent (and f has periodic points of all even periods). Since turbulent maps are known [2] to be topologically semi-conjugate, on some compact invariant subsets, to the shift map on two symbols which is a typical model for chaotic dynamical systems, these maps f^2 (and so f) are chaotic. When we examine closely the above 4 conditions, we find that none is implied by *all* other three (see Figures 1 & 2). So, what do they have in common which make them all so turbulent? In this note, we answer this question by a simple result (Theorem 1) which extends Proposition 3 on page 122 of [2].

Let J be a compact interval in I . If there exist two compact subintervals J_0 and J_1 of J with at most one point in common such that $f(J_0) \cap f(J_1) \supset J_0 \cup J_1$, then we say that f is turbulent on J (and on I) [2]. If there exist two compact subintervals K and L of I with at most one point in common such that f is turbulent on K and on L , then we say that f is doubly turbulent on I .

Theorem 1. *Let f be a continuous map from I into itself and let x_0 be a point in I . Then exactly one of the following holds:*

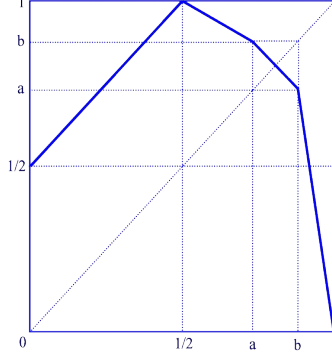


Figure 1: A map satisfying (a), but none of (b), (c) and (d).

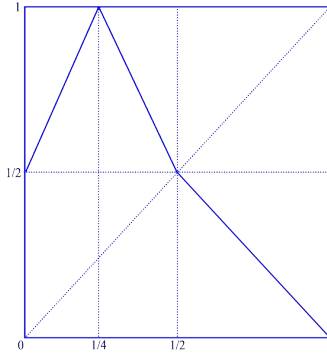


Figure 2: A map satisfying (b), (c) and (d), but not (a).

(A) If there exist a point c in the orbit $O_f(x_0) = \{x_0, f(x_0), f^2(x_0), \dots\}$ of x_0 and an integer $n \geq 2$ such that $f^n(c) \leq c < f(c)$ or $f(c) < c \leq f^n(c)$, then at least one of the following holds:

- (1) There exist a fixed point z of f and a compact subinterval K of I such that (i) $c \in K$, (ii) $f^2(K) \subsetneq K$, (iii) K contains no fixed points of f , and (iv) K and $f(K)$ lie on opposite sides of z , in particular, the iterates of c with respect to f are "jumping" alternately around the fixed point z ;
- (2) f has periodic points of all even periods and f^2 is doubly turbulent.

(B) If $x_i = f^i(x_0)$ for all $i \geq 0$, then either for some $m > 0$ the sequence $\langle x_n \rangle_{n \geq m}$ converges monotonically to a fixed point of f or there exist a fixed point \hat{z} of f and a strictly increasing sequence $0 \leq n_0 < n_1 < n_2 < \dots$ of integers such that if $x_0 < x_1$ (if $x_0 > x_1$ then all inequalities below are reversed) then

$$x_0 < x_1 < \dots < x_{n_0-1} < x_{n_1} < x_{n_1+1} < \dots < x_{n_2-1} < x_{n_3} < x_{n_3+1} < \dots < x_{n_4-1} < \dots < \hat{z} \\ < \dots < x_{n_5-1} < \dots < x_{n_4+1} < x_{n_4} < x_{n_3-1} < \dots < x_{n_2+1} < x_{n_2} < x_{n_1-1} < \dots < x_{n_0+1} < x_{n_0}$$

and if $p = \lim_{i \rightarrow \infty} x_{n_{2i+1}}$ and $q = \lim_{i \rightarrow \infty} x_{n_{2i}}$ then $p \leq \hat{z} \leq q$ and, $f(p) = q$ and $f(q) = p$. In

particular, x_0 is asymptotically periodic of period 1 or 2, i.e., there is a periodic point y of f with $f^2(y) = y$ such that $\lim_{n \rightarrow \infty} |f^n(x_0) - f^n(y)| = 0$.

Proof. If the hypothesis of (A) fails, then it is clear that (B) holds. Now, assume that $f^n(c) \leq c < f(c)$ for some point c in $O_f(x_0)$. If $f(c) < c \leq f^n(c)$, the proof is similar. Let $X = \{f^i(c) : 0 \leq i \leq n-1\}$. Let $a = \max\{x \in X : f(x) > x\}$ and let b be any point in $X \cap [a, f(a)]$ such that $f(b) \leq a$. Then $c \leq a$. Let z be a fixed point of f in $[a, b]$ and let v be a point in $[a, z]$ such that $f(v) = b$. So, $f^2(v) = f(b)$ and $\max\{c, f^2(v)\} \leq a \leq v < z < b = f(v)$. Let $z_0 = \min\{v \leq x \leq z : f^2(x) = x\}$. Then $f(x) > z$ and $f^2(x) < x$ for all $v \leq x < z_0$. We have three cases to consider:

Case 1. If $f^2(x) < z_0$ for all $\min I \leq x \leq v$, then $f^2(x) < z_0 \leq z < f(x)$ for all $\min I \leq x \leq z_0$. Let t be a point in (v, z_0) such that $t > f^2(x)$ for all $\min I \leq x \leq t$. Let $K = [\min I, t]$. Then $c \in [\min I, v] \subset K$, $f^2(K) \subset [\min I, t] \subsetneq K$, K contains no fixed points of f , and K and $f(K)$ lie on opposite sides of z .

Case 2. If the point $d = \max\{\min I \leq x \leq v : f^2(x) = z_0\}$ exists and $\min\{f^2(x) : d \leq x \leq z_0\} = s > d$, then $f(x) > z \geq z_0 > f^2(x) \geq s$ for all $d < x < z_0$. Let \tilde{t} be a point in (v, z_0) such that $\tilde{t} > f^2(x)$ for all $s \leq x \leq v$. Let $K = [s, \tilde{t}]$. Then K contains no fixed points of f , K and $f(K)$ lie on opposite sides of z and $f^2(K) \subset [s, \tilde{t}] \subsetneq K$. Furthermore, for some $2 \leq k \leq n$, $f^{k-1}(c) = b$ and so, $f^k(c) = f(b) = f^2(v) \in f^2(K) \subset K$. Consequently, $f^k(c) \in K$. Since $f(K \cup f(K)) \subset K \cup f(K)$ and $n \geq k$, we have $f^n(c) = f^{n-k}(f^k(c)) \in K \cup f(K)$. Since $f^n(c) (\leq c \leq v) < z$, this forces $f^n(c) \in K$. Since $\tilde{t} \in [s, \tilde{t}] = K$ and $f^n(c) \leq c \leq v < \tilde{t}$, this in turn implies that $c \in K$.

Case 3. If both the point $d = \max\{\min I \leq x \leq v : f^2(x) = z_0\}$ and the point $u_1 = \min\{d \leq x \leq z_0 : f^2(x) = d\}$ exist, then $f(x) > z \geq z_0 > f^2(x)$ on (d, z_0) and $f^2([d, u_1]) \cap f^2([u_1, z_0]) \supset [d, z_0] = [d, u_1] \cup [u_1, z_0]$. In particular, f^2 is turbulent on $[d, z_0] \subset [\min I, z]$. Furthermore, since $u_1 = \min\{d \leq x \leq z_0 : f^2(x) = d\}$, we have $d < f^2(x) < z_0$ on (d, u_1) . Let p_1 be any point in (d, u_1) such that $f^2(p_1) = p_1$. Let $u_2 = \min\{d \leq x \leq p_1 : f^2(x) = u_1\}$. Then $d < (f^2)^2(x) < z_0$ on (d, u_2) . Let p_2 be any point in (d, u_2) such that $(f^2)^2(p_2) = p_2$. Inductively, we obtain points $d < \dots < p_n < u_n < \dots < p_2 < u_2 < p_1 < u_1 < z_0$ such that $u_n = \min\{d \leq x \leq p_{n-1} : (f^2)^{n-1}(x) = u_1\}$, $d < (f^2)^n(x) < z_0$ on (d, u_n) and $(f^2)^n(p_n) = p_n$. Since $f(x) > z \geq z_0$ on (d, z_0) , we have $f^i(p_n) < z_0 < f^j(p_n)$ for all even i and all odd j in $[0, 2n]$. So, each p_n is a period- $(2n)$ point of f . This confirms that f has periodic points of all *even* periods. Finally, since d is the largest point in $[\min I, z_0]$ such that $f^2(d) = z_0$, f must map the endpoints of $[d, z_0]$ into the endpoints of $f([d, z_0])$ and no points x in (d, z_0) can satisfy $f(x) = f(d)$ or $f(x) = f(z_0)$. Consequently, if $f(d) > f(z_0)$ (if $f(d) < f(z_0)$, the proof is similar), then $f([d, z_0]) = [f(z_0), f(d)]$ and, for some $\hat{s} \leq d$, $f((f(z_0), f(d))) = f^2((d, z_0)) = [\hat{s}, z_0] \supset [d, z_0]$. Let e be a point in $(f(z_0), f(d))$ such that $f(e) = d$. Then $f^2([f(z_0), e]) \cap f^2([e, f(d)]) \supset [f(z_0), f(d)] = [f(z_0), e] \cup [e, f(d)]$. Furthermore, if $f(d) = f(z_0)$ and $f([d, z_0]) = [r, f(d)]$ for some point $r > z$ (if $f(d) = f(z_0)$ and $f([d, z_0]) = [f(d), r]$, the proof is similar), then since $f([r, f(d)]) = f^2([d, z_0]) \supset [d, z_0]$, there exists a point u in $[r, f(d)]$ such that $f(u) = d$. Since $f^2([u, f(d)]) \supset f([d, z_0]) = [r, f(d)]$, there exists a point w in $(u, f(d))$ such that $f^2(w) = r$. Therefore, $f^2([u, w]) \cap f^2([w, f(d)]) \supset [r, f(d)] \supset [u, f(d)] = [u, w] \cup [w, f(d)]$. In either case, f^2 is turbulent on $[z, \max I]$. This, combined with the above, shows that f^2 is doubly turbulent on I . \square

In Part (A)(1) of the above result, the compact interval K is not an ordinary one. It is one with the following 4 properties that (i) $c \in K$; (ii) $f^2(K) \subsetneq K$; (iii) K contains no fixed points of f ; and (iv) $f(K) \cap K = \emptyset$. By choosing the appropriate point c , it is the violation of one of these properties that establishes the following result in which (2) and (4) are generalizations of (b) and (d) above respectively.

Corollary 2. *Each of the following statements implies that f has periodic points of all even periods and f^2 is doubly turbulent:*

- (1) *There exist a point c and an odd integer $n > 1$ such that $f^n(c) \leq c < f(c)$ or $f(c) < c \leq f^n(c)$, in particular, f has a periodic point of odd period > 1 ;*
- (2) *The chain recurrent points of f are dense in I and $f^2(a) \neq a$ for some point a in I (recall that a chain recurrent point is a point x which satisfies that for every $\varepsilon > 0$ there exist a finite sequence of points $x_i, 0 \leq i \leq n$ such that $x_0 = x = x_n$ and $|f(x_i) - x_{i+1}| < \varepsilon$ for all $0 \leq i \leq n-1$. Note that if x_0 is a chain recurrent point of f with $f(x_0) < x_0$ ($f(x_0) > x_0$ respectively), then by discussing the three cases similar to those three in the above proof of Theorem 1 with x_0 replacing z_0 , we can obtain (see Lemma 32 on page 150 of [2]) a point c such that $f(c) < c < f^2(c) = x_0$ ($x_0 = f^2(c) > c > f(c)$ respectively));*
- (3) *The ω -limit set $\omega_f(b)$ of some point b in I contains a fixed point z of f and a point $\neq z$;*
- (4) *There is a point in I which is not asymptotically periodic of period 1 or 2 and the set $\{(x, y) : \liminf_{n \rightarrow \infty} |f^n(x) - f^n(y)| = 0\}$ is dense in $I \times I$, in particular, f is densely chaotic;*
- (5) *There is a point c in I such that*

$$\limsup_{n \rightarrow \infty} |f^n(c) - f^{n+1}(c)| > 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} |f^n(c) - f^{n+1}(c)| = 0.$$

The following result can be proved similarly.

Theorem 3. *If there exist a fixed point z of f and a point c of I such that $f(c) < c < z$ or $z < c < f(c)$, then at least one of the following holds:*

- (1) *f has a proper compact interval J in I such that $c \in J$, $f(J) \subsetneq J$ and $z \notin J$;*
- (2) *f is turbulent and has periodic points of all periods.*

Consequently, if (1) there exist a fixed point z of f , a point c of I and an integer $n \geq 2$ such that $f(c) < c < z \leq f^n(c)$ or, $f^n(c) \leq z < c < f(c)$, or $(f(c) - z)/(c - z) > 1$ and $z \in \omega_f(c)$; or (2) the chain recurrent points of f are dense in I and f has at least two fixed points and $f(a) \neq a$ for some point a in I , then f is turbulent and has periodic points of all periods.

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